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# A NOTE ON SIMULTANEOUS RECURRENCE CONDITIONS ON A SET OF DENUMERABLE STOCHASTIC MATRICES

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A note on simultaneous recurrence conditions on a set of denumerable stochastic matrices *

by

A. Federgruen, A. Hordijk ** & H.C. Tijms ***

ABSTRACT

In this paper we consider a set of denumerable stochastic matrices where the parameter set is a compact metric space. We give a number of simultaneous recurrence conditions on the stochastic matrices and establish equivalences between these conditions. The results obtained generalize corresponding results in Markov chain theory to a considerable extent and have applications in stochastic control problems.

KEY WORDS & PHRASES: *compact metric set of denumerable stochastic matrices, simultaneous recurrence conditions, Doeblin condition, scrambling condition, quasi-compactness condition, equivalences*

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# 1. INTRODUCTION

We consider a set $\mathcal{P} = (P(f), f\in F)$ of stochastic matrices $P(f) = (p_{ij}(f))$, $i,j \in I$ having a denumerable state space I where the parameter set F is a compact metric space. Note that, for any $f \in F$, $p_{ij}(f) \geq 0$ and $\sum_{j\in I} p_{ij}(f) = 1$. It is assumed that for any $i,j \in I$ the function $p_{ij}(f)$ is continuous on F. Further, we assume that for any $f \in F$ the stochastic matrix $P(f)$ has no two disjoint closed sets of states.

For any $f \in F$, denote by the stochastic matrix $P^n(f) = (p^n_{ij}(f)$, $i, j \in I$ the n-fold matrix product of $P(f)$ with itself for $n = 1,2,\ldots$ . Note that for any $i,j \in I$ and $n \geq 1$ the function $p^n_{ij}(f)$ is continuous on F. For any $i_0 \in I$, $A \subset I$ and $f \in F$, define the taboo probability

(1) $$t^n_{i_0A}(f) = \sum_{i_1,\ldots,i_n \in I\backslash A} p_{i_0i_1}(f) \ldots p_{i_{n-1}i_n}(f), \quad n = 1,2,\ldots.$$

i.e. $t^n_{iA}(f)$ is the probability that under the stochastic matrix $P(f)$ the first return to the set A takes more than n transitions starting from state i. For any $i \in I$, $A \subset I$ and $f \in F$, define the (possibly infinite) mean recurrence time

(2) $$\mu_{iA}(f) = 1 + \sum_{n=1}^{\infty} t^n_{iA}(f).$$

We write $t^n_{iA}(f) = t_{ij}(f)$ and $\mu_{iA}(f) = \mu_{ij}(f)$ for $A = \{j\}$. Consider now the following simultaneous recurrence conditions on the set $\mathcal{P} = (P(f), f \in F)$.

C1. *There is a finite set* K *and a finite number* B *such that*

$$\mu_{iK}(f) \leq B \text{ for all } i \in I \text{ and } f \in F.$$

C2. *There is a finite set* K, *an integer* $\nu \geq 1$ *and a number* $\rho > 0$ *such that*

$$\sum_{j\in K} p^{\nu}_{ij}(f) \geq \rho \text{ for all } i \in I \text{ and } f \in F.$$

C3. *There is an integer* $\nu \geq 1$ *and a number* $\rho \quad 0$ *such that*

$$\sum_{j \in I} \min[p^{\nu}_{i_1 j}(f), p^{\nu}_{i_2 j}(f)] \geq \rho \text{ for all } i_1, i_2 \in I \text{ and } f \in F.$$

C4. *There is an integer* $\nu \geq 1$ *and a number* $\rho > 0$ *such that for any* $f \in F$ *a probability distribution* $\{\pi_j(f), j \in I\}$ *(say) exists for which*

$$\left| \sum_{j \in A} p^{n}_{ij}(f) - \sum_{j \in A} \pi_j(f) \right| \leq (1-\rho)^{[n/\nu]} \text{ for all } i \in I,\ A \subset I \text{ and } n \geq 1. \tag{3}$$

*where* $[x]$ *denotes the largest integer less than or equal to* $x$.

C5. *For any* $f \in F$ *there is a probability distribution* $\{\pi_j(f), j \in I\}$ *such that*

$$p^{n}_{ij}(f) \to \pi_j(f) \text{ uniformly in } (i,f) \in I \times F \text{ as } n \to \infty \text{ for any } j \in I. \tag{4}$$

C6. *There is a finite number* $B$ *such that for any* $f \in F$ *a state* $s_f$ *exists for which*

$$\mu_{is_f}(f) \leq B \text{ for all } i \in I.$$

C7. *There is a finite set* $K$ *and a finite number* $B$ *such that for any* $f \in F$ *a state* $s_f \in K$ *exists for which*

$$\mu_{is_f}(f) \leq B \text{ for all } i \in I.$$

C8. *There is an integer* $\nu \geq 1$ *and a number* $\rho > 0$ *such that for any* $f \in F$ *a state* $s_f$ *exists for which*

$$p^{\nu}_{is_f}(f) \geq \rho \text{ for all } i \in I.$$

C9. *There is a finite set* K, *an integer* $\nu \geq 1$ *and a number* $\rho > 0$ *such that for any* $f \in F$ *a state* $s_f \in K$ *exists for which*

$$p^{\nu}_{is_f}(f) \geq \rho \text{ for all } i \in I.$$

We note that in C4 the condition $\sum_{j \in I} |p^n_{ij}(f) - \pi_j(f)| \leq 2(1-\rho)^{[n/\nu]}$ for all $i \in I$, $f \in F$ and $n \geq 1$ may be equivalently stated instead of (3).

The following two theorems were obtained in [4] and [2] (cf. also [3]).

THEOREM 1. *The conditions* C1 *and* C2 *are equivalent.*

THEOREM 2.

(i) *If the stochastic matrix* P(f) *is aperiodic for each* $f \in F$, *then the condition* C2 *implies the condition* C3.

(ii) *The condition* C3 *implies the condition* C4.

In this paper we shall prove the following additional relations.

THEOREM 3.

(i) *The condition* C5 *implies both condition* C2 *and* C9.

(ii) *The conditions* C3, C4, C5, C8 *and* C9 *are equivalent.*

THEOREM 4.

(i) *The condition* C2 *implies the condition* C7.

(ii) *The condition* C6 *implies the condition* C7.

(iii) *The conditions* C1, C2, C6 *and* C7 *are equivalent.*

(iv) *If the stochastic matrix* P(f) *is aperiodic for each* $f \in F$, *then the conditions* C1-C9 *are equivalent.*

In case the set $\mathcal{P}$ consists of a single stochastic matrix, the conditions C2, C3 and C4-C5 are known in Markov chain theory as the Doeblin, the scrambling and the quasi-compactness (or strong ergodicity) condition respectively, and the above equivalences may be found, albeit in a scattered way, in the literature, cf. p.197 in [1], p.142 in [5], p.226 in [6] and p.185 in [7]. The above results generalize the corresponding results in Markov chain theory to a considerable extent and have applications amongst others in semi-Markov decision problems, cf. [2] - [4].

## 2. PROOFS

In this section we prove the Theorems 3 and 4.

*Proof of Theorem 3.* (i) Suppose that condition C5 holds. Since for any $i,j \in I$ and $n \geq 1$ the function $p_{ij}^{n}(f)$ is continuous on F, it follows from (4) that for any $j \in I$ the function $\pi_j(f)$ is continuous in $f \in F$. Now, let $\{K_n, n=1,2,...\}$ be a sequence of finite sets $K_n \subset I$ such that $K_{n+1} \supseteq K_n$ for all $n \geq 1$ and $\lim_{n\to\infty} K_n = I$. Let $a_n(f) = \sum_{j\in K_n} \pi_j(f)$ for $n \geq 1$ and $f \in F$. Then the function $a_n(f)$ is continuous in $f \in F$ for any $n \geq 1$ and, moreover, for any $f \in F$ we have $a_{n+1}(f) \geq a_n(f)$ for all $n \geq 1$ and $\lim_{n\to\infty} a_n(f) = 1$. Now, since F is compact, we have by Theorem 7.13 in [8] that $a_n(f)$ converges to 1 uniformly in $f \in F$ as $n \to \infty$. Hence for each $\varepsilon > 0$ there is a finite integer n such that $a_n(f) \geq 1-\varepsilon$ for all $f \in F$. This shows that we can find a finite set K and a number $\delta > 0$ such that

$$\sum_{j\in K} \pi_j(f) \geq \delta \quad \text{for all } f \in F. \tag{5}$$

By (4) and the finiteness of K, we can find an integer $\nu \geq 1$ such that $p_{ij}^{\nu}(f) \geq \pi_j(f) - \delta/2|K|$ for all $i \in I$, $f \in F$ and $j \in K$ where $|K|$ denotes the number of states in K. Together this inequality and (5) imply condition C2. Further we get from (5) that for any $f \in F$ there is a state $s_f$ such that $\pi_{s_f} \geq \delta/|K|$ and so $p_{is_f}^{\nu}(f) \geq \delta/2|K|$ for all $i \in I$ and $f \in F$. This inequality verifies condition C9 which completes the proof of part (i).

(ii) Since C9 implies C8 and in its turn C8 implies C3 and since C4 implies C5, this part follows by using part (ii) of Theorem 2 and part (i) of Theorem 3.

*Proof of Theorem 4.* To prove this Theorem, we shall use a classical perturbation of the stochastic matrices P(f), $f \in F$. Fix any number $\tau$ with $0 < \tau \leq 1$ and let $\bar{P} = (\bar{P}(f), f \in F)$ be the set of stochastic matrices $\bar{P}(f) = (\bar{p}_{ij}(f))$, $i, j \in I$ such that for any $f \in F$ and $i,j \in I$

$$\bar{p}_{ij}(f) = \begin{cases} \tau p_{ij}(f) & \text{for } j \neq i \\ 1 - \tau + \tau p_{ii}(f) & \text{for } j = i \end{cases}$$

Note that, by $p_{ii}(f) \geq 1 - \tau > 0$ for all $i \in I$ and $f \in F$, the stochastic matrix $\bar{P}(f)$ is *aperiodic* for all $f \in F$. Also note that for any $i,j \in I$ the function $\bar{p}_{ij}(f)$ is continuous in $f \in F$ and for any $f \in F$, the stochastic matrix $\bar{P}(f)$ has no two disjoint closed sets. Define for the stochastic matrices $\bar{P}(f)$ the taboo probabilities $\bar{t}^n_{iA}(f)$ and the mean recurrence times $\bar{\mu}_{iA}(f)$ as in (1) and (2). By induction on n, it is straightforward to verify that for any $f \in F$

(6) $$\bar{t}^n_{ij}(f) = \sum_{k=0}^{n} \binom{n}{k}(1-\tau)^{n-k}\tau^k t^k_{ij}(f) \quad \text{for all } n = 0,1,\ldots \text{ and } i,j \in I \text{ with } i \neq j,$$

where $\bar{t}^0_{ij}(f) = t^0_{ij}(f) = 1$. From the relations (2) and (6) we get

(7) $$\bar{\mu}_{ij}(f) = \frac{\mu_{ij}(f)}{\tau} \quad \text{for all } i,j \in I \text{ with } i \neq j \text{ and } f \in F.$$

We note that this relation in intuitively clear by a direct probabilistic interpretation.

We now prove (i). Suppose that the condition C2 holds with triple $(K,\nu,\rho)$. Then, by $\bar{p}_{ij}(f) \geq \tau p_{ij}(f)$ for all $i,j \in I$ and $f \in F$, we have

$$\sum_{j \in K} \bar{p}_{ij}(f) \geq \tau^\nu \sum_{j \in K} p^\nu_{ij}(f) \geq \tau^\nu \rho \quad \text{for all } i \in I \text{ and } f \in F.$$

Hence the condition C2 applies to the set $\bar{\mathcal{P}} = (P(f), f \in F)$. Moreover we have that any stochastic matrix $\bar{P}(f)$, $f \in F$ is aperiodic. Now, by Theorem 2 and part (i) of Theorem 3, it follows that condition C9 applies to the set $\bar{\mathcal{P}}$. Since condition C9 implies C7, we have that condition C7 applies to the set $\bar{\mathcal{P}}$. Now, by invoking (7), it follows that the condition C7 holds for the set $\mathcal{P} = (P(f), f \in F)$ as was to be proved.

Next we prove (ii). Suppose that condition C6 holds. Then, by invoking again (7), we have that condition C6 applies to the set $\bar{\mathcal{P}}$. Hence there is a finite number B such that for any $f \in F$ there exists a state

$s_f$ such that

(8) $$\bar{\mu}_{is_f}(f) = 1 + \sum_{n=1}^{\infty} \bar{t}^n_{is_f}(f) \le B \text{ for all } i \in I.$$

Fix now $0 < \gamma < 1$. Since for any $f \in F$ and $i \in I$ the taboo probability $\bar{t}^n_{is_f}(f)$ is non-increasing in n, it follows that there is an integer $N \ge 1$ such that

(9) $$\bar{t}^N_{is_f}(f) \le \gamma \quad \text{for all } i \in I \text{ and } f \in F.$$

(Supposing the contrary to (9) gives a contradiction with (8)). Together the inequality (9) and the fact that $\bar{p}_{kk}(f) \ge 1 - \tau$ for all $k \in I$ and $f \in F$ imply

$$\bar{p}^N_{is_f}(f) \ge (1-\tau)^{N-1}(1-\gamma) \quad \text{for all } i \in I \text{ and } f \in F.$$

This shows that condition C8 applies to the set $\bar{P}$. Next, by part (ii) of Theorem 3, condition C9 applies to the set $\bar{P}$. Since C9 implies C7, it follows that condition C7 applies to the set $\bar{P}$. Now by invoking again (7) we have that condition C7 holds for the stochastic matrices P(f), $f \in F$ as was to be verified.

We obtain part (iii) of the Theorem by noting that C7 trivially implies both C1 and C6 and using Theorem 1 and the parts (i) - (ii) of Theorem 4. Finally, part (iv) of the Theorem is an immediate consequence of the Theorems 2-3 and part (iii) of Theorem 4.

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